

Content Knowledge for Teaching

What Makes It Special?

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This article reports the authors' efforts to develop a practice-based theory of content knowledge for teaching built on Shulman's (1986) notion of pedagogical content knowledge. As the concept of pedagogical content knowledge caught on, it was in need of theoretical development, analytic clarification, and empirical testing. The purpose of the study was to investigate the nature of professionally oriented subject matter knowledge in mathematics by studying actual mathematics teaching and identifying mathematical knowledge *for teaching* based on analyses of the mathematical problems that arise in teaching. In conjunction, measures of mathematical knowledge for teaching were developed. These lines of research indicate at least two empirically discernable subdomains within pedagogical content knowledge (*knowledge of content and students* and *knowledge of content and teaching*) and an important subdomain of "pure" content knowledge unique to the work of teaching, *specialized content knowledge*, which is distinct from the *common content knowledge* needed by teachers and nonteachers alike. The article concludes with a discussion of the next steps needed to develop a useful theory of content knowledge for teaching.

Keywords: *mathematics; teacher knowledge; pedagogical content knowledge*

Most people would agree that an understanding of content matters for teaching. Yet, what constitutes understanding of the content is only loosely defined. In the mid-1980s, a major breakthrough initiated a new wave of interest in the conceptualization of teacher content knowledge. Lee Shulman (1986) and his colleagues proposed a special domain of teacher knowledge that they termed pedagogical content knowledge. What provoked broad interest was the suggestion that there is content knowledge unique to teaching—a kind of subject-matter-specific professional knowledge. The continuing appeal of the notion of pedagogical content knowledge is that it bridges content knowledge and the practice of teaching. However, after two decades of work, this bridge between knowledge and practice was still inadequately understood and the coherent theoretical framework Shulman (1986, p. 9) called for remained underdeveloped. This article builds on the promise of pedagogical content knowledge, reporting new progress on the nature of content knowledge for teaching.

Although the term pedagogical content knowledge is widely used, its potential has been only thinly developed. Many seem to assume that its nature and content are obvious. Yet what is meant by pedagogical content

knowledge is underspecified. The term has lacked definition and empirical foundation, limiting its usefulness.

Throughout the past 20 years, for example, researchers have used pedagogical content knowledge to refer to a wide range of aspects of subject matter knowledge and the teaching of subject matter and, indeed, have used it differently across—and even within—subject areas. Besides differences in the breadth of what the term includes, there have been significant differences in how the term is used to relate content knowledge to the practice of teaching. Frequent, for example, are broad claims about what teachers need to know. Such statements are often more normative than empirical. Only a few studies have tested whether there are, indeed, distinct bodies of identifiable content knowledge that matter for teaching.

Authors' Note: The research reported in this article was supported by grants from the National Science Foundation (Grants REC 0126237, REC-0207649, EHR-0233456, and EHR-0335411) and the Spencer Foundation (MG 199800202). The authors thank Hyman Bass, Heather Hill, Laurie Sleep, Suzanne Wilson, and members of the Mathematics Teaching and Learning to Teach Project and of the Learning Mathematics for Teaching Project for their help in developing aspects of this article. Errors are the responsibility of the authors.

Without this empirical testing, the ideas are bound to play a limited role in improving teaching and learning—in revamping the curriculum for teacher content preparation, in informing policies about certification and professional development, and in furthering our understanding of the relationships among teacher knowledge, teaching, and student learning. Without this empirical testing, the ideas remain, as they were 20 years ago, promising hypotheses based on logical and ad hoc arguments about the content believed to be necessary for teachers.

For the last 15 years, the work of the Mathematics Teaching and Learning to Teach Project and of the Learning Mathematics for Teaching Project has focused both on the teaching of mathematics and on the mathematics used in teaching. Although the context of our work has been mathematics, we have sought to contribute to a broader discussion by researchers in different school subjects. To consider the knowledge that teaching entails, we began by investigating what teaching itself demands. Instead of reasoning from the school curriculum to a list of topics teachers must know, we developed an empirical approach to understanding the content knowledge needed for teaching. The first project focused on the *work* teachers do in teaching mathematics. The authors and their colleagues used studies of teaching practice to analyze the mathematical demands of teaching and, based on these analyses, developed a set of testable hypotheses about the nature of mathematical knowledge for teaching. In a related line of work, the second project developed survey measures of content knowledge for teaching mathematics. The measures provided a way to investigate the nature, the role, and the importance of different types of mathematical knowledge for teaching.

In particular, these studies have led us to hypothesize some refinements to the popular concept of pedagogical content knowledge and to the broader concept of content knowledge for teaching. In this article, we focus on the work of teaching in order to frame our conceptualization of the mathematical knowledge and skill needed by teachers. We identify and define two empirically detectable subdomains of pedagogical content knowledge. In addition, and to our surprise, we have begun to uncover and articulate a less recognized domain of content knowledge for teaching that is not contained in pedagogical content knowledge, but yet—we hypothesize—is essential to effective teaching. We refer to this as *specialized content knowledge*. These possible refinements to the map of teacher content knowledge are the subject of this article.

Because the work of Shulman and his colleagues is foundational, we begin by reviewing the problem they framed, the progress they made, and the questions that remained unanswered. We use this discussion to clarify

the problems of definition, empirical basis, and practical utility that our work addresses. We then turn to mathematics in particular, describe work on the problem of identifying mathematical knowledge for teaching, and report on refinements to the categories of mathematical knowledge for teaching. The article concludes with an appraisal of next steps in developing a useful theory of content knowledge for teaching.

Content Knowledge and Its Role in Defining Teaching as a Profession

A central contribution of Shulman and his colleagues was to reframe the study of teacher knowledge in ways that attend to the role of content in teaching. This was a radical departure from research of the day, which focused almost exclusively on general aspects of teaching. Subject matter was little more than context. Although earlier studies were conducted in classrooms where mathematics, reading, or other subjects were taught, attention to the subject itself and to the role it played in teaching or teacher thinking was less prominent. In fact, so little attention was devoted to examining content and its role in instruction that Shulman dubbed this the “missing paradigm” in research on teaching and teacher knowledge (1986).

A second contribution of Shulman and his colleagues was to represent content understanding as a special kind of technical knowledge key to the profession of teaching. In the late 1980s, they conducted case studies of beginning high school teachers as part of their research in the Knowledge Growth in Teaching project. Participants were recent graduates with strong subject matter preparation in mathematics, science, English literature, and history. By examining these novices in the process of learning to teach, the group sought to investigate how strong subject matter preparation translated into the knowledge needed for teaching that subject. Deliberately working across subjects provided a comparative basis for examining more general characteristics of the knowledge that the teachers used in their practice.

A closely related purpose was to draw from these categories of teacher knowledge to inform the development of a National Board system for the certification of teachers that would “focus upon the teacher’s ability to reason about teaching and to teach specific topics, and to base his or her actions on premises that can bear the scrutiny of the professional community” (Shulman, 1987, p. 20). Attention to certification was deliberately geared toward informing debates about what constitutes professional expertise and what such expertise implies

Figure 1
Shulman's Major Categories of Teacher Knowledge

- General pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter
 - Knowledge of learners and their characteristics
 - Knowledge of educational contexts, ranging from workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures
 - Knowledge of educational ends, purposes, and values, and their philosophical and historical grounds
 - Content knowledge
 - Curriculum knowledge, with particular grasp of the materials and programs that serve as "tools of the trade" for teachers
 - Pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding
- (Shulman, 1987, p. 8)

for teacher preparation and for policy decisions. In particular, Shulman was concerned with prevailing conceptions of teacher competency, which focused on generic teaching behaviors. He argued that "the currently incomplete and trivial definitions of teaching held by the policy community comprise a far greater danger to good education than does a more serious attempt to formulate the knowledge base" (Shulman, 1987, p. 20). Implicit in such comments is the argument that high-quality instruction requires a sophisticated, professional knowledge that goes beyond simple rules such as how long to wait for students to respond.

To characterize professional knowledge for teaching, Shulman and his colleagues developed typologies. Although the specific boundaries and names of categories varied across publications, one of the more complete articulations is reproduced in Figure 1.

These categories were intended to highlight the important role of content knowledge and to situate content-based knowledge in the larger landscape of professional knowledge for teaching. The first four categories address general dimensions of teacher knowledge that were the mainstay of teacher education programs at the time. They were not the main focus of Shulman's work. Instead, they functioned as placeholders in a broader conception of teacher knowledge that emphasized content knowledge. At the same time, however, Shulman made clear that these general categories were crucial and that an emphasis placed on content dimensions of teacher knowledge was not intended to minimize the importance of pedagogical understanding and skill: Shulman (1986) argued that "mere content knowledge is likely to be as useless pedagogically as content-free skill" (p. 8).

The remaining three categories define content-specific dimensions and together comprise what Shulman referred to as the missing paradigm in research on teaching—"a blind spot with respect to content that characterizes most research on teaching, and as a consequence, most of our state-level programs of teacher evaluation and teacher certification" (1986, pp. 7-8). The first, *content knowledge*, includes knowledge of the subject and its organizing structures (see also Grossman, Wilson, & Shulman, 1989; Wilson, Shulman, & Richert, 1987). Drawing on Schwab (1961/1978), Shulman (1986) argued that knowing a subject for teaching requires more than knowing its facts and concepts. Teachers must also understand the organizing principles and structures and the rules for establishing what is legitimate to do and say in a field. The teacher need not only understand *that* something is so; the teacher must further understand *why* it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened or denied. Moreover, we expect the teacher to understand why a particular topic is particularly central to a discipline whereas another may be somewhat peripheral. (p. 9)

The second category, *curricular knowledge*, is "represented by the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances" (p. 10). In addition, Shulman pointed to two other dimensions of curricular knowledge that are important for teaching, aspects that he labeled *lateral curriculum knowledge* and *vertical curriculum knowledge*. Lateral knowledge relates knowledge of the curriculum being taught to the curriculum that students are learning in other classes (in other subject areas). Vertical knowledge includes "familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school, and the materials that embody them" (Shulman, 1986, p. 10).

The last, and arguably most influential, of the three content-related categories was the new concept of *pedagogical content knowledge*. Shulman (1986) defined pedagogical content knowledge as comprising:

The most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the most useful ways of representing and formulating the subject that make it comprehensible to others. . . . Pedagogical

content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. (p. 9)

The claim for pedagogical content knowledge was founded on observations that effective teachers in the Knowledge Growth in Teaching study represented key ideas using metaphors, diagrams, and explanations that were at once attuned to students' learning and to the integrity of the subject matter (see also Carlsen, 1988; Grossman, 1990; Marks, 1990; Wilson, 1988; Wilson et al., 1987; Wineburg, 1990). Some representations are especially powerful; others, although technically correct, do not open the ideas effectively to learners.

A second important idea is that representations of the subject are informed by content-specific knowledge of student conceptions. A focus on conceptions, and in many cases a particular interest in student misconceptions, acknowledges that accounting for how students understand a content domain is a key feature of the work of teaching that content. Grossman (1990) pointed out that these ideas

are inherent in Dewey's admonition that teachers must learn to "psychologize" their subject matter for teaching, to rethink disciplinary topics to make them more accessible to students. . . . Teachers must draw upon both their knowledge of subject matter to select appropriate topics and their knowledge of students' prior knowledge and conceptions to formulate appropriate and provocative representations of the content to be learned. (p. 8)

As a concept, pedagogical content knowledge, with its focus on representations and conceptions/misconceptions, broadened ideas about how knowledge might matter to teaching, suggesting that it is not only knowledge of content, on the one hand, and knowledge of pedagogy, on the other hand, but also a kind of amalgam of knowledge of content and pedagogy that is central to the knowledge needed for teaching. In Shulman's (1987) words, "Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from the pedagogue" (p. 8).

Over the course of Shulman and his colleagues' work, the categories for teacher knowledge underwent a number of revisions. The researchers were clear that they saw their understanding of teacher knowledge as incomplete and distinctions and labels as provisional. They appear to have seen the value in these distinctions as

heuristic, as a tool for helping the field to identify distinctions in teacher knowledge that could matter for effective teaching.

Shulman and his colleagues did not seek to build a list or catalogue of what teachers need to know in any particular subject area. Instead, their work sought to provide a conceptual orientation and a set of analytic distinctions that would focus the attention of the research and policy communities on the nature and types of knowledge needed for teaching a subject. In drawing attention to the missing paradigm, or the virtual absence of research focused directly on teacher content knowledge, Shulman and his colleagues defined a perspective that highlighted the content-intensive nature of teaching. However, they also sought to specify the ways in which content knowledge for teaching is distinct from disciplinary content knowledge. This had important implications for informing an emerging argument that teaching is professional work with its own unique professional knowledge base.

Testing Shulman's Hypothesis About Content Knowledge and Pedagogical Content Knowledge

There was immediate and widespread interest in the ideas presented by Shulman and his colleagues. In the two decades since these ideas were first presented, Shulman's presidential address (1986) and the related *Harvard Education Review* article (1987) have been cited in more than 1,200 refereed journal articles. This interest has been sustained with no less than 50 citations to these two articles in every year since 1990. Perhaps most remarkable is the reach of this work, with citations appearing in 125 different journals, in professions ranging from law to nursing to business, and regarding knowledge for teaching students preschool through doctoral studies. Much of the interest has focused directly on pedagogical content knowledge. Thousands of articles, book chapters, and reports use or claim to study the notion of pedagogical content knowledge, in a wide variety of subject areas: science, mathematics, social studies, English, physical education, communication, religion, chemistry, engineering, music, special education, English language learning, higher education, and others. Such studies show no signs of abating. Rarely does an idea catch on so widely.

But how has the field taken up the idea of pedagogical content knowledge? What have we learned, and what do we yet need to understand?

Much of the work that followed in the wake of Shulman's proposals showed how teachers' orientations to content influenced the ways in which they taught that content. Grossman (1990) showed how teachers' orientations to literature shaped the ways in which they approached texts with their students. Wilson and Wineburg (1988) described how social studies teachers' disciplinary backgrounds—in political science, anthropology, sociology—shaped the ways in which they represented historical knowledge for high school students. And Ball (1990) introduced the phrase “knowledge *about* mathematics” to contrast with “knowledge *of* mathematics” and to highlight the nature of knowledge in the discipline—where it comes from, how it changes, and how truth is established. In science education, study of the “nature of science” showed that specific orientations are aligned with distinct subdisciplines and significantly influence the teaching carried out in classrooms. For instance, teachers trained in biology teach physics courses differently than do teachers trained in physics or in chemistry.

A second line of work—some of it predating the introduction of pedagogical content knowledge—has contributed to our understanding of the knowledge teachers need about common conceptions and misconceptions that students bring to the classroom or develop as they learn a subject. For instance, Wineburg's (1990) analysis of students' natural efforts to understand motives and explanations for past events can be at cross-purposes with the special nature of historical understanding. Smith and Anderson (1984) showed that children's conceptions of food and eating persistently interfered with their learning about the process of photosynthesis as the means by which plants make their own food. Likewise, in the Cognitively Guided Instruction project, researchers found that students overgeneralize from experiences with problems in which the equals sign acts as a signal to compute (as it does in many programming languages) (Carpenter, Franke, & Levi, 2003; Carpenter & Levi, 2000). In other words, given the problem $5 + 7 = _ + 8$, students are likely to answer 12 or 20, where the equal sign is interpreted as a signal to add. Fueled by developments in cognitive science and by increased attention to the role of prior knowledge in theories of learning, investigations into what teachers need to know about students' conceptions and misconceptions of particular subject matter have flourished. This line of research elaborates the concept of pedagogical content knowledge by showing the special ways in which teaching demands a simultaneous integration of key ideas in the content with ways in which students apprehend them.

In another line of work provoked by Shulman's call to attend to content, researchers documented the lack of teachers' content and pedagogical content knowledge. In mathematics, Ball (1988) developed interview questions that revealed, on the one hand, the inadequacies of teachers' and prospective teachers' knowledge of important mathematics needed for teaching and, on the other hand, how much there was to understand. Ma (1999) used the tasks developed by Ball for her studies to elaborate more fully the special nature of the content knowledge needed for teaching that is beyond simply “knowing” the content. Finding the perimeter of a rectangle is different from analyzing a student's unanticipated generalization about the relationship between perimeter and area. The first requires only knowing how to calculate perimeter; the second requires an ability to think flexibly about perimeter to analyze another's claim. Borko et al. (1992) described the case of a middle school student teacher, Ms. Daniels, who was asked by a child to explain why the invert-and-multiply algorithm for dividing fractions works. Despite having taken 2 years of calculus, a course in proof, a course in modern algebra, and four computer science courses *and* being able to divide fractions herself, Ms. Daniels was nonetheless unable to provide a correct representation for division of fractions or to explain why the invert-and-multiply algorithm works. In addition, examination of the instances when Ms. Daniels did successfully teach for conceptual understanding revealed the central importance of using appropriate representations that made the content comprehensible to students.

The notion of pedagogical content knowledge has permeated scholarship on teaching and teacher education but has done so unevenly across fields. Interestingly, our survey of the literature shows that roughly one fourth of the articles about pedagogical content knowledge are in science education, with slightly fewer in mathematics education. However, it is the breadth of literature on pedagogical content knowledge that highlights the term's heuristic value as a way of conceptualizing teacher knowledge. In physical education, the term helps to distinguish a teacher's own proficiency in a skill area (e.g., throwing a ball or dribbling) from the explicit knowledge of the skill that is needed in order to teach it to students (Chen, 2002; Rovegno, Chen, & Todorovich, 2003). There is a growing recognition that teaching reading requires a detailed knowledge of text, language, and reading process that goes beyond just being able to decode and comprehend text proficiently (Hapgood, Palincsar, Kucan, Gelpi-Lomangino, & Khasnabis, 2005; Moats, 1999; Phelps, 2005; Phelps & Schilling, 2004).

Still, however, the field has made little progress on Shulman's initial charge: to develop a coherent theoretical framework for content knowledge for teaching. The ideas remain theoretically scattered, lacking clear definition. Because researchers tend to specialize in a single subject, much of the work has unfolded in roughly parallel but independent strands. Often it is unclear how ideas in one subject area relate to another or even whether findings within the same subject take similar or different views of teacher subject matter knowledge. Somewhat ironically, nearly one third of the articles that cite pedagogical content knowledge do so without direct attention to a specific content area, instead making general claims about teacher knowledge, teacher education, or policy. Scholars have used the concept of pedagogical content knowledge as though its theoretical foundations, conceptual distinctions, and empirical testing were already well defined and universally understood.

Particularly striking is the lack of definition of key terms. Pedagogical content knowledge is often not clearly distinguished from other forms of teacher knowledge, sometimes referring to something that is simply content knowledge and sometimes to something that is largely pedagogical skill. Most definitions are perfunctory and often broadly conceived. This appears to be the case across all subject areas. For example, pedagogical content knowledge has been defined as "the intersection of knowledge of the subject with knowledge of teaching and learning" (Niess, 2005, p. 510) or as "that domain of teachers' knowledge that combines subject matter knowledge and knowledge of pedagogy" (Lowery, 2002, p. 69). In even broader terms, pedagogical content knowledge is defined simply as "the product of transforming subject matter into a form that will facilitate student learning" (de Berg & Greive, 1999, p. 20). Although these and a host of other short definitions capture the general idea of pedagogical content knowledge as a domain that combines the subject with teaching, they are broad enough to include nearly any package of teacher knowledge and beliefs.

A definition's brevity, however, is not the only factor that contributes to a lack of clarity over what might count as pedagogical content knowledge. More careful and detailed definitions still leave unclear where the boundary is between pedagogical content knowledge and other forms of teacher knowledge. For example, Magnusson, Krajcik, and Borko (1999) defined the construct as follows.

Pedagogical content knowledge is a teacher's understanding of how to help students understand specific subject matter. It includes knowledge of how particular subject matter topics, problems, and issues can be organized,

represented and adapted to the diverse interests and abilities of learners, and then presented for instruction. . . . The defining feature of pedagogical content knowledge is its conceptualization as the result of a *transformation* of knowledge from other domains. (p. 96)

When defined in these ways, pedagogical content knowledge begins to look as though it includes almost everything a teacher might know in teaching a particular topic, obscuring distinctions between teacher actions, reasoning, beliefs, and knowledge.

We argue that the power of the idea, launched by Shulman and his colleagues, that teaching requires a special kind of content knowledge is worth our collective investment and cultivation. That teaching demands content knowledge is obvious; policy makers are eager to set requirements based on commonsense notions of content knowledge. Scholars can help to specify the nature of content knowledge needed, but providing this specification demands that we use greater precision about the concepts and methods involved. Our aim in this article is to describe how we have approached this problem in the context of mathematics and what we are learning about the nature of the content knowledge needed for teaching.

Our Approach to Studying Mathematical Knowledge for Teaching

In the past, a focus on what teachers need to know has led to a set of positions, each related to principled arguments about what teachers should know. The prevailing view is that teachers need to know whatever mathematics is in the curriculum plus some additional number of years of further study in college mathematics. A second hypothesis is that teachers need to know the curriculum, but "deeper," plus some amount of pedagogical content knowledge. In both cases, it is unclear what exactly it is that makes up the extra knowledge of mathematics.

A more focused question is this: What do teachers need to know and be able to do in order to teach effectively? Or, what does effective teaching require in terms of content understanding? This places the emphasis on the use of knowledge in and for teaching rather than on teachers themselves. These are centrally important questions that could be investigated in numerous ways—by examining the curriculum and standards for which teachers are responsible (or the tests their students must be prepared to pass), by asking expert mathematicians and mathematics educators to identify the core mathematical ideas and skills that teachers should have (CBMS, 2001), or by reviewing research on students'

learning to ascertain those aspects of mathematics with which learners have difficulty (Stylianides & Ball, 2004). Our research group chose a different approach, one that might be characterized as working from the bottom up, beginning with practice. Because it seemed obvious that teachers need to know the topics and procedures that they teach—primes, equivalent fractions, functions, translations and rotations, factoring, and so on—we decided to focus on how teachers need to know that content. In addition, we wanted to determine what else teachers need to know about mathematics and how and where teachers might use such mathematical knowledge in practice.

Hence, we decided to focus on the work of teaching. What do teachers need to do in teaching mathematics—by virtue of being responsible for the teaching and learning of content—and how does this work demand mathematical reasoning, insight, understanding, and skill? Instead of starting with the curriculum, or with standards for student learning, we study the work that teaching entails. In other words, although we examine particular teachers and students at given moments in time, our focus is on what this actual instruction suggests for a detailed job description. What fundamental activities are demanded by the broad aims of developing a classroom in which mathematics is treated with integrity, students' ideas are taken seriously, and mathematical work is a collective as well as an individual endeavor? We seek to unearth the ways in which mathematics is involved in contending with the regular day-to-day, moment-to-moment demands of teaching.

Our analyses lay the foundation for a practice-based theory of mathematical knowledge for teaching (Ball & Bass, 2003b). We see this approach as a kind of job analysis, similar to analyses done of other mathematically intensive occupations that range from nursing, banking, and engineering (Hoyle, Noss, & Pozzi, 2001; Kent, Noss, Guile, Hoyle, & Bakker, 2007; Noss & Hoyle, 1996) to carpentry and waiting tables (Milroy, 1992).

By “mathematical knowledge for teaching,” we mean the mathematical knowledge needed to carry out the work of teaching mathematics. Important to note here is that our definition begins with teaching, not teachers. It is concerned with the tasks involved in teaching and the mathematical demands of these tasks. Because teaching involves showing students how to solve problems, answering students' questions, and checking students' work, it demands an understanding of the content of the school curriculum. Beyond these obvious tasks, we seek to identify other aspects of the work and to analyze what these reveal about the content demands of teaching.

We continue to approach the problem in two ways. First, we conduct extensive qualitative analyses of teaching practice. Second, we design measures of mathematical knowledge for teaching based on hypotheses formulated from our qualitative studies. We briefly describe these two lines of work and their intersection.

The following questions guide our qualitative analyses:

1. What are the recurrent tasks and problems of teaching mathematics? What do teachers do as they teach mathematics?
2. What mathematical knowledge, skills, and sensibilities are required to manage these tasks?

By “teaching,” we mean everything that teachers must do to support the learning of their students. Clearly we mean the interactive work of teaching lessons in classrooms and all the tasks that arise in the course of that work. But we also mean planning for those lessons, evaluating students' work, writing and grading assessments, explaining the classwork to parents, making and managing homework, attending to concerns for equity, and dealing with the building principal who has strong views about the math curriculum. Each of these tasks, and many others as well, involve knowledge of mathematical ideas, skills of mathematical reasoning, fluency with examples and terms, and thoughtfulness about the nature of mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001).

Central to the qualitative work has been a large longitudinal National Science Foundation–funded database, documenting an entire year of the mathematics teaching in a third grade public school classroom during 1989–1990. The records collected across that year include videotapes and audiotapes of the classroom lessons, transcripts, copies of students' written class work, homework, and quizzes as well as the teacher's plans, notes, and reflections. A second resource has been the wide range of experiences and disciplinary backgrounds of the members of our research group. A third major resource has been a set of analytic tools we have developed for coordinating mathematical and pedagogical perspectives (Thames, 2008).

We have been studying not only specific episodes but also instruction over time, considering the work of developing both mathematics and students across the school year (Ball & Bass, 2000, 2003a). What sort of larger picture of a mathematical topic and its associated practices is needed for teaching over time? How do students' ideas and practices develop, and what does this imply about the mathematical work of teachers? In addition to using

this extensive set of records, we work also with other collections we have assembled over the last decade. These collections, like our original one, typically include videotapes of classroom teaching, copies of students' work and of teachers' notes, and curriculum materials from which the teacher is teaching.

By coordinating mathematical and pedagogical perspectives in the analysis of these detailed records of practice, we seek to develop a practice-based theory of mathematical knowledge as it is entailed by and used in teaching (Ball, 1999; Thames, 2008). On the one hand, the generality of our results may be limited because our data are limited to only a few classrooms all situated in the U.S. context. On the other hand, our results are likely to be broadly applicable because our conception of the work of teaching is based, not on a particular approach to teaching, but on identifying fundamental tasks entailed in teaching.

As a complement to our ongoing qualitative analyses of teaching and as a fortuitous result of our engagement in a large study of comprehensive school reform models (the Study of Instructional Improvement; www.sii.soe.umich.edu), we began to develop and validate survey measures of mathematical knowledge for teaching (Ball, Hill, & Bass, 2005; Hill, Ball, & Schilling, 2004; Hill, Rowan, & Ball, 2005). To do so, we again engage multidisciplinary teams to draft, refine, and critique questions (Bass & Lewis, 2005). In this work, we write items in different categories and test the questions with large groups of teachers (Hill & Ball, 2004). Analytic techniques, such as factor analysis, provide a basis for testing our assumptions about the structure of mathematical knowledge for teaching and help us to refine the categories and the measures. Although these analyses are ongoing, we see persuasive evidence that the mathematical knowledge needed for teaching is multidimensional. That is, general mathematical ability does not fully account for the knowledge and skills entailed in teaching mathematics (Hill et al., 2004).

Like Shulman, we think it important to identify, isolate, and measure the knowledge and skill distinctive of teaching and essential to establishing its status as a professional activity—even though we recognize that in actual teaching such boundaries can seem artificial. Approaching the problem by analyzing teaching practice and developing instruments to test the ideas, we are able to do the kind of discovery and refinement called for by Shulman in 1987—we are able to fill in some of the rudimentary “periodic table” of teacher knowledge.

In the subsequent sections, we describe our research group's current hypotheses about the structure and domains of mathematical knowledge for teaching and summarize evidence for these domains. We then return to the notion of pedagogical content knowledge and discuss

the relationship of our work to that body of research. First, though, we provide an illustration of the kind of knowledge that has surfaced from our analyses of teaching. Perhaps most interesting to us has been evidence that teaching may require a specialized form of pure subject matter knowledge—“pure” because it is not mixed with knowledge of students or pedagogy and is thus distinct from the pedagogical content knowledge identified by Shulman and his colleagues and “specialized” because it is not needed or used in settings other than mathematics teaching. This uniqueness is what makes this content knowledge special.

An Example of What Makes Mathematical Knowledge for Teaching Special

Our analyses of teachers' practice reveal that the mathematical demands of teaching are substantial. The mathematical knowledge needed for teaching is not less than that needed by other adults. In fact, knowledge for teaching must be detailed in ways unnecessary for everyday functioning. In short, a teacher needs to know more, and different, mathematics—not less. To better explain what we mean by this, we offer an example based on a simple subtraction computation:

$$\begin{array}{r} 307 \\ - 168 \\ \hline \end{array}$$

Most readers will know an algorithm to produce the answer 139, such as the following:

$$\begin{array}{r} \overset{2}{3} \overset{9}{0} 7 \\ - 168 \\ \hline 139 \end{array}$$

We start with this pure computational task because teachers who teach subtraction must be able to perform this calculation themselves. This is mathematical knowledge that others commonly hold, because this knowledge is used in a wide range of settings. However, being able to carry out this procedure is necessary, but not sufficient, for teaching it. We next take the example further into the work of teaching.

Many third graders struggle with the subtraction algorithm, often making errors. One common error is the following:

$$\begin{array}{r} 307 \\ - 168 \\ \hline 261 \end{array}$$

A teacher must be able to spot that 261 is incorrect. This does not require any special knowledge to do: Anyone who can solve the problem above can readily see this. However, teaching involves more than identifying an incorrect answer. Skillful teaching requires being able to size up the source of a mathematical error. Moreover, this is work that teachers must do rapidly, often on the fly, because in a classroom, students cannot wait as a teacher puzzles over the mathematics himself. Here, for example, a student has, in each column, calculated the difference between the two digits, or subtracted the smaller digit from the larger one. A teacher who is mystified about what could have produced 261 as an answer will arguably move more slowly and with less precision to help correct the student's problem. Consider another error that teachers may confront when teaching this subtraction problem:

$$\begin{array}{r} 307 \\ -168 \\ \hline 169 \end{array}$$

What mathematical steps would produce this error? In this case, in contrast to the first example, the solution is based on "borrowing" 1 from the hundreds column, "carrying the 1" to the ones place, and subtracting 8 from 17, yielding 9. The process continues by "bringing down" the 6 and calculating $2 - 1 = 1$. Teachers need to be able to perform this kind of mathematical error analysis efficiently and fluently. Error analysis is a common practice among mathematicians in the course of their own work; the task in teaching differs only in that it focuses on the errors produced by learners.

These two errors stem from different difficulties with the algorithm for subtracting multidigit numbers. In the first, the student considered the difference between digits with no thought to the relationships among columns. In the second, the student attempted to regroup the number but without attention to the value of the places. Seeing both answers as simply wrong does not equip a teacher with the detailed mathematical understanding required for a skillful treatment of the problems these students face.

Analysis such as this are characteristic of the distinctive work teachers do, and they require a kind of mathematical reasoning that most adults do not need on a regular basis. Although mathematicians engage in analyses of error, often of failed proofs, the analysis used to uncover a student error appears to be related to, but not the same as, other error analyses in the discipline. Furthermore, whereas teachers must process such analyses fluently, no

demand exists for mathematicians to conduct their work quickly.

It is also common in instruction for students to produce nonstandard approaches that are unfamiliar to the teacher. For instance, what mathematical issues confront a teacher if a student asserts that she would "take 8 away from both the top and the bottom," yielding the easier problem:

$$\begin{array}{r} 299 \\ -160 \\ \hline \end{array}$$

Is it legitimate to do this? Why? Would it work in general? Is it easier for some numbers and harder for others? How might you describe the method the student is using and how would you justify it mathematically? Being able to engage in this sort of mathematical inner dialogue and to provide mathematically sound answers to these questions is a crucial foundation for determining what to *do* in teaching this mathematics.

Teachers confront all kinds of student solutions. They have to figure out what students have done, whether the thinking is mathematically correct for the problem, and whether the approach would work in general. Consider the following three executions of our original subtraction problem. What is going on mathematically in each case?

307	307	307
-168	-168	-168
<hr style="width: 100%;"/>	<hr style="width: 100%;"/>	<hr style="width: 100%;"/>
-1	139	2
-60		30
<hr style="width: 100%;"/>		<hr style="width: 100%;"/>
139		107
		<hr style="width: 100%;"/>
		139

These examples are all correct and could be generalized in plausible ways, but figuring this out is not a straightforward task for those who only know how to do the subtraction as they themselves learned it.

Interpreting student error and evaluating alternative algorithms is not all that teachers do. Teaching also involves explaining procedures. For example, for the subtraction algorithm, one could give a set of procedural directions. A teacher might say, "Cross out the 3, put a 2, put a 1 on top of the 0, cross out the 1 and the 0 and put a 9, and then put a 1 by the 7; now subtract." However, this procedure is specific to this problem: It does not generalize to, for instance, $314 - 161$, where one only "crosses out" and "puts" once, not twice. It also does nothing to show how the procedure works. Teachers

must know rationales for procedures, meanings for terms, and explanations for concepts. Teachers need effective ways of representing the meaning of the subtraction algorithm—not just to confirm the answer but to show what the steps of the procedure mean and why they make sense. Our point here is not about what teachers need to teach to children but about what teachers themselves must know and be able to do to carry out that teaching.

How might teachers explain the meaning of the subtraction algorithm to students? One possibility is to use money as a model. To represent $307 - 168$, what money would be needed? First, a teacher needs to recognize that not all U.S. coins come in denominations in the base-ten numeration system. In making change for 68 cents, you would be likely to use two quarters, a dime, a nickel, and three pennies. But the base-ten system does not use 25 or 5 as units; instead it uses the decimal units—100, 10, 1. Representing 68 cents with 6 dimes and 8 pennies is obviously possible in money but is not the most typical or efficient choice given the coins we have. Giving students 3 U.S. dollars and 7 pennies and asking them to take away \$1.68 does not lead readily to regrouping \$3.07 into 2 dollars, 9 dimes, and 17 pennies, which would be necessary to use money to represent the regrouping central to the conventional subtraction algorithm. Furthermore, carrying out the regrouping of \$3.07 in a manner that fits the standard algorithm requires 10 dimes, not 9. What might a different model make visible? For instance, money requires “trading” one dime for 10 pennies, whereas straws rubber banded into groups of 10 can be used to model the processes of grouping 10 ones into one 10 and ungrouping one 10 into 10 ones.

Teaching also involves considering what numbers are strategic to use in an example. The numbers 307 and 168 may not be ideal choices to make visible the conceptual structure of the algorithm. Should the numerical examples require two regroupings, as in this case, or should examples be sequenced from ones requiring no regrouping to ones that require several? What about the role of zeros at different points in the procedure? Should the example include zeros—or perhaps not at first? Questions such as these, as well as those posed in the discussion above, require mathematical reasoning and insight, crucial to teaching, yet foreign to most well-educated adults. This is what we mean by the special mathematical demands of teaching mathematics.

Our study of the mathematical demands of teaching has yielded a wealth of tasks that require mathematical knowledge and skill. What caught us by surprise, however, was how much special mathematical knowledge

was required, even in many everyday tasks of teaching—assigning student work, listening to student talk, grading or commenting on student work. Despite the fact that these tasks are done with and for students, close analysis revealed how intensively mathematical the tasks were. We were surprised to see that many of the component tasks of teaching require mathematical knowledge apart from knowledge of students or teaching. For instance, deciding whether a method or procedure would work in general requires mathematical knowledge and skill, not knowledge of students or teaching. It is a form of mathematical problem solving used in the work of teaching. Likewise, determining the validity of a mathematical argument, or selecting a mathematically appropriate representation, requires mathematical knowledge and skill important for teaching yet not entailing knowledge of students or teaching. In our research we began to notice how rarely these mathematical demands could be addressed with mathematical knowledge learned in university mathematics courses. We began to hypothesize that there were aspects of subject matter knowledge—in addition to pedagogical content knowledge—that need to be uncovered, mapped, organized, and included in mathematics courses for teachers.

Three points are central to our argument. First, much of the work of teaching is mathematical in nature, with significant mathematical demands. Although the mathematical tasks we have identified would inform teachers’ choices and actions with students, these tasks can also be seen as illustrating the special mathematical thinking that teachers must do and understand in order to teach mathematics. These tasks require significant mathematical knowledge, skill, habits of mind, and insight. Although our examples are drawn from the context of teaching, the *mathematical* knowledge needed to engage them stands on its own as a domain of understanding, disposition, and skill needed by teachers for their work.

A second point is that the mathematical knowledge we have identified here has a relevance to teaching that is often missing from discussions about the mathematics needed by teachers. By identifying mathematics in relation to specific tasks in which teachers engage, we establish its relevance to what teachers do. Part of the value of the notion of pedagogical content knowledge is that it offers a way to build bridges between the academic world of disciplinary knowledge and the practice world of teaching; it does so by identifying amalgam knowledge that combines the knowing of content with the knowing of students and pedagogy. Our practice-based conceptualization of content knowledge for teaching provides an additional way of building bridges between these two worlds; it does so by defining knowledge in

broad terms, including skill, habits of mind, and insight, and by framing knowledge in terms of its use—in terms of particular tasks of teaching.

Finally, we suspect that many of these insights extend to the knowledge teachers need in other subjects as well. What might these insights mean in the context of teaching history, or biology, or music?

In our analyses of the mathematical work involved in teaching mathematics, we noticed that the nature of that mathematical knowledge and skill seemed itself to be of different types. We hypothesized that teachers' opportunities to learn mathematics for teaching could be better tuned if we could identify those types more clearly. If mathematical knowledge required for teaching is indeed multidimensional, then professional education could be organized to help teachers learn the range of knowledge and skill they need in focused ways. If, however, the mathematical knowledge required for teaching is basically the same as general mathematical ability, then discriminating professional learning opportunities would be unnecessary. Based on our analysis of the mathematical demands of teaching, we hypothesized that Shulman's content knowledge could be subdivided into CCK and specialized content knowledge and his pedagogical content knowledge could be divided into knowledge of content and students and knowledge of content and teaching. Turning back to the results of our studies, in the next section we define and illustrate each of these subdomains.

Mathematical Knowledge for Teaching and Its Structure

In analyzing the mathematical demands of teaching, we seek to identify mathematical knowledge that is demanded by the work teachers do. To pursue this, we define the mathematical knowledge we are studying as mathematical knowledge "entailed by teaching"—in other words, mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students. To avoid a strictly reductionist and utilitarian perspective, however, we seek a generous conception of "need" that allows for the perspective, habits of mind, and appreciation that matter for effective teaching of the discipline.

The first domain represents the first step in the example above: simply calculating an answer or, more generally, correctly solving mathematics problems. We call this *common content knowledge (CCK)* and define it as the mathematical knowledge and skill used in settings other than teaching. Teachers need to know the material they teach; they must recognize when their students give wrong answers or when the textbook gives an inaccurate

definition. When teachers write on the board, they need to use terms and notation correctly. In short, they must be able to do the work that they assign their students. But some of this requires mathematical knowledge and skill that others have as well—thus, it is not special to the work of teaching. By "common," however, we do not mean to suggest that everyone has this knowledge. Rather, we mean to indicate that this is knowledge of a kind used in a wide variety of settings—in other words, not unique to teaching.

When we analyzed videos of teaching, it was obvious that such knowledge is essential. When a teacher mispronounced terms, made calculation errors, or got stuck trying to solve a problem at the board, instruction suffered and valuable time was lost. In mapping out the mathematical knowledge needed by teachers, we found that an understanding of the mathematics in the student curriculum plays a critical role in planning and carrying out instruction.

Additional evidence for common content knowledge comes from our work to develop instruments for measuring mathematical knowledge for teaching. We pose questions such as, "What is a number that lies between 1.1 and 1.11?" We ask questions that require knowing that a square is a rectangle, that $0/7$ is 0, and that the diagonals of a parallelogram are not necessarily perpendicular. These are not specialized understandings but are questions that typically would be answerable by others who know mathematics. Often, as shown in Figure 2, we couch the problem in the context of teaching to point out where in the activity of teaching the use of such common knowledge might arise.

Figure 2
Statements about 0

Ms. Dominguez was working with a new textbook and she noticed that it gave more attention to the number 0 than her old book did. She came across a page that asked students to determine if a few statements about 0 were true or false. Which statement(s) should she recognize as true?

	Yes	No
a) 0 is an even number.	1	2
b) 0 is not really a number. It is a placeholder in writing big numbers.	1	2
c) The number 8 can be written as 008.	1	2

The activity of looking over textbooks requires, among other things, basic competence with the content. Knowing which statements are true in Figure 2 is common mathematical knowledge that is not likely to be unique to teachers.

Figure 3
Mathematical Tasks of Teaching

Presenting mathematical ideas
Responding to students' "why" questions
Finding an example to make a specific mathematical point
Recognizing what is involved in using a particular representation
Linking representations to underlying ideas and to other representations
Connecting a topic being taught to topics from prior or future years
Explaining mathematical goals and purposes to parents
Appraising and adapting the mathematical content of textbooks
Modifying tasks to be either easier or harder
Evaluating the plausibility of students' claims (often quickly)
Giving or evaluating mathematical explanations
Choosing and developing useable definitions
Using mathematical notation and language and critiquing its use
Asking productive mathematical questions
Selecting representations for particular purposes
Inspecting equivalencies

The second domain, *specialized content knowledge (SCK)*, is the mathematical knowledge and skill unique to teaching. This is the domain in which we have become particularly interested. Close examination reveals that SCK is mathematical knowledge not typically needed for purposes other than teaching. In looking for patterns in student errors or in sizing up whether a nonstandard approach would work in general, as in our subtraction example, teachers have to do a kind of mathematical work that others do not. This work involves an uncanny kind of unpacking of mathematics that is not needed—or even desirable—in settings other than teaching. Many of the everyday tasks of teaching are distinctive to this special work (Figure 3).

Each of these tasks is something teachers routinely do. Taken together, these tasks demand unique mathematical understanding and reasoning. Teaching requires knowledge beyond that being taught to students. For instance, it requires understanding different interpretations of the operations in ways that students need not explicitly distinguish; it requires appreciating the difference between “take-away” and “comparison” models of subtraction and between “measurement” and “partitive” models of division. Consider, for instance, the problem in Figure 4.

The mathematics of this problem can be rather challenging. The first word problem is division by 2 rather than by $\frac{1}{2}$; the second is multiplication by 2 rather than

Figure 4
Story Problem That Represents $1\frac{1}{4}$ divided by $\frac{1}{2}$

Which of the following story problems can be used to represent $1\frac{1}{4}$ divided by $\frac{1}{2}$?

	Yes	No
a) You want to split $1\frac{1}{4}$ pies evenly between two families. How much should each family get?	1	2
b) You have \$1.25 and may soon double your money. How much money would you end up with?	1	2
c) You are making some homemade taffy and the recipe calls for $1\frac{1}{4}$ cups of butter. How many sticks of butter (each stick = $\frac{1}{2}$ cup) will you need?	1	2

division by $\frac{1}{2}$ (a subtle yet important point for teaching this content); and the third correctly fits the calculation—using a measurement meaning of division. The important point here, though, is that figuring out which story problems fit with which calculations, and vice versa, is a task engaged in teaching this content, not something done in solving problems with this content.

Teaching involves the use of decompressed mathematical knowledge that might be taught directly to students as they develop understanding. However, with students the goal is to develop fluency with compressed mathematical knowledge. In the end, learners should be able to use sophisticated mathematical ideas and procedures. Teachers, however, must hold unpacked mathematical knowledge because teaching involves making features of particular content visible to and learnable by students. Teaching about place value, for example, requires understanding the place-value system in a self-conscious way that goes beyond the kind of tacit understanding of place value needed by most people. Teachers, however, must be able to talk explicitly about how mathematical language is used (e.g., how the mathematical meaning of *edge* is different from the everyday reference to the edge of a table); how to choose, make, and use mathematical representations effectively (e.g., recognizing advantages and disadvantages of using rectangles or circles to compare fractions); and how to explain and justify one's mathematical ideas (e.g., why you invert and multiply to divide fractions). All of these are examples of ways in which teachers work with mathematics in its decompressed or unpacked form.

Some might wonder whether this decompressed knowledge is equivalent to conceptual understanding. They might ask whether we would not want all learners to understand content in such ways. Our answer is no. What we are describing is more than a solid grasp of the

material. We do not hold as a goal that every learner should be able to select examples with pedagogically strategic intent, to identify and distinguish the complete range of different situations modeled by $38 \div 4$, or to analyze common errors.

The mathematical demands of teaching require specialized mathematical knowledge not needed in other settings. Accountants have to calculate and reconcile numbers and engineers have to mathematically model properties of materials, but neither group needs to explain why, when you multiply by 10, you “add a zero.” In developing survey questions to measure such knowledge, we ask, for example, whether an unusual method proposed by a student would work in general, which statement best explains why we find common denominators when adding fractions, and which of a set of given drawings could be used to represent 2 divided by $\frac{2}{3}$. These and questions like them are the daily fare of teaching. The demands of the work of teaching mathematics create the need for such a body of mathematical knowledge specialized to teaching.

The third domain, *knowledge of content and students (KCS)*, is knowledge that combines knowing about students and knowing about mathematics. Teachers must anticipate what students are likely to think and what they will find confusing. When choosing an example, teachers need to predict what students will find interesting and motivating. When assigning a task, teachers need to anticipate what students are likely to do with it and whether they will find it easy or hard. Teachers must also be able to hear and interpret students’ emerging and incomplete thinking as expressed in the ways that pupils use language. Each of these tasks requires an interaction between specific mathematical understanding and familiarity with students and their mathematical thinking.

Central to these tasks is knowledge of common student conceptions and misconceptions about particular mathematical content. For instance, in the subtraction example, knowing that students often “subtract up” when confronted with a problem such as $307 - 168$ means that a teacher who has seen this happen and knows that it is a common student response is able to recognize it without extensive mathematical analysis or probing. In other words, recognizing a wrong answer is common content knowledge (CCK), whereas sizing up the nature of an error, especially an unfamiliar error, typically requires nimbleness in thinking about numbers, attention to patterns, and flexible thinking about meaning in ways that are distinctive of specialized content knowledge (SCK). In contrast, familiarity with common errors and deciding which of several errors students are most likely to make are examples of knowledge of content and students (KCS).

Many demands of teaching require knowledge at the intersection of content and students. In developing an instrument to measure such knowledge, we ask questions, for example, about the kinds of shapes young students are likely to identify as triangles, the likelihood that they may write 405 for 45, and problems where confusion between area and perimeter lead to erroneous answers. We also ask questions that require interpretation of students’ emerging and inchoate thinking, that present the thinking or expressions typical of particular learners, or that demand sensitivity to what is likely to be easy or challenging.

Many of our ideas in this area draw from the literature on student thinking: for example, van Hiele’s studies of levels of the development in representing two-dimensional figures (Burger & Shaughnessy, 1986; Crowley, 1987), CGI researchers’ documentation of common misinterpretations of the equal sign (Carpenter et al., 2003; Carpenter & Levi, 2000) or that subtraction problems involving comparison are harder for students than take-away problems (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998), or Philipp, Cabral, and Schappelle’s (2005) observation that students misappropriate the subtraction language of take-away when representing fractions, causing them to confound what is left with what is removed. In each case, knowledge of students and content is an amalgam, involving a particular mathematical idea or procedure and familiarity with what students often think or do.

The last domain, *knowledge of content and teaching (KCT)*, combines knowing about teaching and knowing about mathematics. Many of the mathematical tasks of teaching require a mathematical knowledge of the design of instruction. Teachers sequence particular content for instruction. They choose which examples to start with and which examples to use to take students deeper into the content. Teachers evaluate the instructional advantages and disadvantages of representations used to teach a specific idea and identify what different methods and procedures afford instructionally. Each of these tasks requires an interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning.

Consider for a moment the need to make instructional decisions about which student contributions to pursue and which to ignore or save for a later time. During a classroom discussion, a teacher must decide when to pause for more clarification, when to use a student’s remark to make a mathematical point, and when to ask a new question or pose a new task to further students’ learning. Each of these decisions requires coordination between the mathematics at stake and the instructional options and purposes at play.

One example of KCT would be knowing different instructionally viable models for place value, knowing what each can reveal about the subtraction algorithm, and knowing how to deploy them effectively. What does money afford instructionally for a particular subtraction problem and how is this different from what coffee stirrers bundled with rubber bands would afford? What about base-ten blocks or “unifix” cubes? Each of these can correctly represent subtraction of multidigit numbers, but each represents different aspects of the content that make a difference at different points in students’ learning. Each model also requires different care in use in order to make the mathematical issues salient and usable by students (Cohen, 2005). Knowing how these differences matter for the development of the topic is part of what we call knowledge of content and teaching.

The demands of teaching require knowledge at the intersection of content and teaching. In developing an instrument to measure such knowledge, we ask questions about whether a tape measure would be good for teaching place value, about choosing examples for simplifying radicals for the purpose of discussing multiple strategies, or about sequencing subtraction problems with and without regrouping for instruction. We also ask questions about how language and metaphors can assist and confound student learning—the way language about borrowing or canceling may interfere with understanding of the mathematical principles underlying the subtraction algorithm or the solving of algebraic equations. In each of these examples, knowledge of teaching and content is an amalgam, involving a particular mathematical idea or procedure and familiarity with pedagogical principles for teaching that particular content.

Building a Map of Usable Professional Knowledge of Subject Matter

Several issues about our proposed categories are worth addressing—their relationship to pedagogical content knowledge, the special nature of specialized content knowledge, our use of teaching as a basis for defining the domains, and problems with the categories that need to be addressed.

From our definitions and examples it should be evident that this work may be understood as elaborating on, not replacing, the construct of pedagogical content knowledge. For instance, the last two domains—knowledge of content and students and knowledge of content and teaching—coincide with the two central dimensions of pedagogical content knowledge identified by Shulman (1986): “the conceptions and preconceptions that

students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons” and “the ways of representing and formulating the subject that make it comprehensible to others” (p. 9).

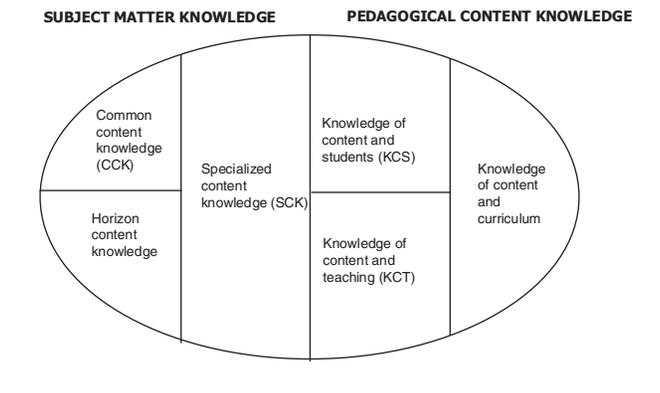
However, we also see our work as developing in more detail the fundamentals of *subject matter knowledge for teaching* by establishing a practice-based conceptualization of it, by elaborating subdomains, and by measuring and validating knowledge of those domains.

We have been most struck by the relatively uncharted arena of mathematical knowledge necessary for teaching the subject that is not intertwined with knowledge of pedagogy, students, curriculum, or other noncontent domains. What distinguishes this sort of mathematical knowledge from other knowledge of mathematics is that it is subject matter knowledge needed by teachers for specific tasks of teaching, such as those in Figure 3, but still clearly subject matter knowledge. These tasks of teaching depend on mathematical knowledge, and, significantly, they have aspects that do not depend on knowledge of students or of teaching. These tasks require knowing how knowledge is generated and structured in the discipline and how such considerations matter in teaching, such as extending procedures and concepts of whole-number computation to the context of rational numbers in ways that preserve properties and meaning. These tasks also require a host of other mathematical knowledge and skills—knowledge and skills not typically taught to teachers in the course of their formal mathematics preparation.

Where, for example, do teachers develop explicit and fluent use of mathematical notation? Where do they learn to inspect definitions and to establish the equivalence of alternative definitions for a given concept? Where do they learn definitions for fractions and compare their utility? Where do they learn what constitutes a good mathematical explanation? Do they learn why 1 is not considered prime or how and why the long division algorithm works? Teachers must know these sorts of things and engage in these mathematical practices themselves when teaching. Explicit knowledge and skills in these areas are vital for teaching.

To represent our current hypotheses, we propose a diagram as a refinement to Shulman’s categories. Figure 5 shows the correspondence between our current map of the domain of content knowledge for teaching and two of Shulman’s (1986) initial categories: subject matter knowledge and pedagogical content knowledge. We have provisionally placed Shulman’s third category, curricular knowledge, within pedagogical content knowledge. This is consistent with later publications from members of

Figure 5
Domains of Mathematical Knowledge for Teaching



Shulman's research team (Grossman, 1990). We are not yet sure whether this may be a part of our category of knowledge of content and teaching or whether it may run across the several categories or be a category in its own right. We also provisionally include a third category within subject matter knowledge, what we call "horizon knowledge" (Ball, 1993). Horizon knowledge is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum. First-grade teachers, for example, may need to know how the mathematics they teach is related to the mathematics students will learn in third grade to be able to set the mathematical foundation for what will come later. It also includes the vision useful in seeing connections to much later mathematical ideas. Having this sort of knowledge of the mathematical horizon can help in making decisions about how, for example, to talk about the number line. Likewise third graders appreciate that the number line they know will soon "fill in" with more and more numbers? And might it matter how a teacher's choices anticipate or distort that later development? Again we are not sure whether this category is part of subject matter knowledge or whether it may run across the other categories. We hope to explore these ideas theoretically, empirically, and also pragmatically as the ideas are used in teacher education or in the development of curriculum materials for use in professional development.

Our current empirical results, based on our factor analyses, suggest it is likely that content knowledge for teaching is multidimensional (Hill et al., 2004; Schilling, in press). Whether these categories, as we propose them here, are the right ones is not most important. Likely they are not. Our current categories will continue to need

refinement and revision. We next highlight three specific problems of our work to date.

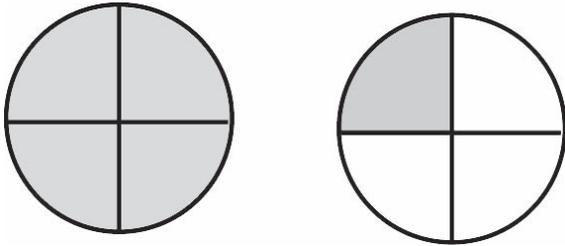
The first problem grows from a strength of the work: Our theory is framed in relation to practice. Although this orientation is intended to increase the likelihood that the knowledge identified is relevant to practice, it also brings in some of the natural messiness and variability of teaching and learning. As we ask about the situations that arise in teaching that require teachers to use mathematics, we find that some situations can be managed using different kinds of knowledge. Consider the example of analyzing a student error. A teacher might figure out what went wrong by analyzing the error mathematically. What steps were taken? What assumptions made? But another teacher might figure it out because she has seen students do this before with this particular type of problem. The first teacher is using specialized content knowledge, whereas the second is using knowledge of content and students.

Two additional problems emerge from the first. Despite our expressed intention to focus on knowledge use, our categories may seem static. Ultimately, we are interested in how teachers reason about and deploy mathematical ideas in their work. We are interested in skills, habits, sensibilities, and judgments as well as knowledge. We want to understand the mathematical reasoning that underlies the decisions and moves made in teaching. The questions we pose in our measures of mathematical knowledge for teaching are designed to situate knowledge in the context of its use, but how such knowledge is actually used and what features of pedagogical thinking shape its use remain tacit and unexamined. How to capture the common and specialized aspects of teacher thinking, as well as how different categories of knowledge come into play in the course of teaching, needs to be addressed more effectively in this work.

Related to this is a boundary problem: It is not always easy to discern where one of our categories divides from the next, and this affects the precision (or lack thereof) of our definitions. We define *common content knowledge* as the mathematical knowledge known in common with others who know and use mathematics, but we do not find that this term always communicates well what we mean. Consequently, although the distinction may be compelling as a heuristic, it can be difficult to discern common from specialized knowledge in particular cases. Take, for instance, the problem of what fraction represents the shaded portion of the two circles shaded in Figure 6.

Is the knowledge that this is $\frac{5}{8}$ of 2 common? Or is it specialized? We tend to think that this kind of detailed knowledge of fractions and their correspondence to a

Figure 6
Representations of $\frac{5}{8}$ of 2



particular representation is specialized knowledge; it is hard to think of others who use this knowledge in their day-to-day work. But perhaps there are others who rely on such detailed and unpacked knowledge of fractions in their work as well. Similarly, it can be difficult at times to discriminate specialized content knowledge from knowledge of content and students—for example, consider what is involved in selecting a numerical example to investigate students' understanding of decimal numbers. The shifts that occur across the four domains, for example, ordering a list of decimals (CCK), generating a list to be ordered that would reveal key mathematical issues (SCK), recognizing which decimals would cause students the most difficulty (KCS), and deciding what to do about their difficulties (KCT), are important yet subtle. That we are able to work empirically as well as conceptually helps us to refine our categories; still, we recognize the problems of definition and precision exhibited by our current formulation.

Finally, we need to understand better the extent to which our formulation of mathematical knowledge for teaching is culturally specific (Cole, 2008; Delaney, 2008) or dependent on teaching styles. We do not think of the knowledge we have been identifying as being closely tied to a particular view of reform or a particular approach to teaching. For instance, interpreting students' thinking, whether in a whole-class discussion or on written homework or a quiz, is an essential part of effectively engaging students in the learning of subject matter. Explaining mathematical ideas is central to teaching, whatever the approach or style. Writing assessment questions, drawing a clear diagram, choosing a counterexample—each of these is a core task of teaching. Still, although our analyses are designed to consider fundamental tasks of teaching content, the particular sample of data we use clearly influences what we do and do not see, and the question of its limitations remains an empirical question.

Conclusion

Teachers must know the subject they teach. Indeed, there may be nothing more foundational to teacher competency. The reason is simple: Teachers who do not themselves know a subject well are not likely to have the knowledge they need to help students learn this content. At the same time, however, just knowing a subject well may not be sufficient for teaching. One need only sit in a classroom for a few minutes to notice that the mathematics that teachers work with in instruction is not the same mathematics taught and learned in college classes. In addition, teachers need to know mathematics in ways useful for, among other things, making mathematical sense of student work and choosing powerful ways of representing the subject so that it is understandable to students. It seems unlikely that just knowing more advanced math will satisfy all of the content demands of teaching. In fact, elementary teachers' mathematics course attainment does not predict their students' achievement gains (National Mathematics Advisory Panel, 2008). What seem most important are knowing and being able to use the mathematics required inside the work of teaching.

Unfortunately, subject matter courses in teacher preparation programs tend to be academic in both the best and worst sense of the word, scholarly and irrelevant, either way remote from classroom teaching. Disciplinary knowledge has the tendency to be oriented in directions other than teaching, toward the discipline—history courses toward knowledge and methods for doing history and science courses toward knowledge and methods for doing science. Although there are exceptions, the overwhelming majority of subject matter courses for teachers, and teacher education courses in general, are viewed by teachers, policy makers, and society at large as having little bearing on the day-to-day realities of teaching and little effect on the improvement of teaching and learning. This is the problem that Shulman and his colleagues addressed in the late 1980s.

In this article, we argue that the issues identified by Shulman and his colleagues more than two decades ago are key to research on teaching and teacher education. Content knowledge is immensely important to teaching and its improvement. Instead of taking pedagogical content knowledge as given, however, we argue that there is a need to carefully map it and measure it. This includes the need to better explicate how this knowledge is used in teaching effectively.

Why are new categories useful? Three reasons capture our current thinking about the usefulness of refining the conceptual map of the content knowledge for teachers.

First, in studying the relationships between teachers' content knowledge and their students' achievement, it would be useful to ascertain whether there are aspects of teachers' content knowledge that predict student achievement more than others. If, for instance, teachers' specialized content knowledge is the greatest predictor of students' achievement, this might direct our efforts in ways different than if advanced content knowledge has the largest effect. However, such studies are sorely missing. Second, it could be useful to study whether and how different approaches to teacher development have different effects on particular aspects of teachers' pedagogical content knowledge. Third, and closely related, a clearer sense of the categories of content knowledge for teaching might inform the design of support materials for teachers as well as teacher education and professional development. Indeed, it might clarify a curriculum for the content preparation of teachers that is professionally based—both distinctive, substantial and fundamentally tied to professional practice and to the knowledge and skill demanded by the work.

The work reported here takes Shulman's charge seriously. It is rooted in attention to the demands of practice to consider what mathematics arises in the work that teachers do. Our work tests these ideas by developing instruments to measure this knowledge, by using the results to inform our understanding of a map of teacher content knowledge, and by tying this knowledge to its use in practice. That there is a domain of content knowledge unique to the work of teaching is a hypothesis that has already developed. However, the notion of specialized content knowledge is in need of further work in order to understand the most important dimensions of teachers' professional knowledge. Doing so with care promises to have significant implications for understanding teaching and for improving the content preparation of teachers.

References

- Ball, D. L. (1988). *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education*. Unpublished doctoral dissertation, Michigan State University, East Lansing.
- Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal*, 90, 449-466.
- Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal*, 93(4), 373-397.
- Ball, D. L. (1999). Crossing boundaries to examine the mathematics entailed in elementary teaching. In T. Lam (Ed.), *Contemporary mathematics* (pp. 15-36). Providence, RI: American Mathematical Society.
- Ball, D. L., & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. C. Phillips (Ed.), *Constructivism in education: Opinions and second opinions on controversial issues. Yearbook of the National Society for the Study of Education* (pp. 193-224). Chicago: University of Chicago Press.
- Ball, D. L., & Bass, H. (2003a). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 27-44). Reston, VA: National Council of Teachers of Mathematics.
- Ball, D. L., & Bass, H. (2003b). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group* (pp. 3-14). Edmonton, Alberta, Canada: Canadian Mathematics Education Study Group (Groupe Canadien d'étude en didactique des mathématiques).
- Ball, D. L., Hill, H. H., & Bass, H. (2005, Fall). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, pp. 14-46.
- Bass, H., & Lewis, J. (2005, April). *What's in collaborative work? Mathematicians and educators developing measures of mathematical knowledge for teaching*. Paper presented at the Annual Meeting of the American Educational Research Association, Montreal, Quebec, Canada.
- Borko, H., Eisenhart, M., Brown, C., Underhill, R., Jones, D., & Agard, P. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education*, 23(3), 194-222.
- Burger, W. F., & Shaughnessy, J. M. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education*, 17, 31-48.
- Carlsen, W. (1988). *The effects of science teacher subject-matter knowledge on teacher questioning and classroom discourse*. Unpublished doctoral dissertation, Stanford University, Palo Alto, CA.
- Carpenter, T. P., Franke, M. L., Jacobs, V. R., Fennema, E., & Empson, S. B. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for Research in Mathematics Education*, 29(1), 3-20.
- Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.
- Carpenter, T. P., & Levi, L. (2000). *Developing conceptions of algebraic reasoning in the primary grades* (research report). Madison: University of Wisconsin-Madison, National Center for Improving Student Learning and Achievement in Mathematics and Science.
- Chen, W. (2002). Six expert and student teachers' views and implementation of constructivist teaching using a movement approach to physical education. *Elementary School Journal*, 102(3), 255-272.
- Cohen, R. B. (2005). *Examining the work of constructing a representational context in elementary mathematics teaching*. Unpublished doctoral dissertation, University of Michigan, Ann Arbor.
- Cole, Y. (2008). *Mathematical knowledge for teaching in Ghana*. Manuscript in preparation.
- Conference Board of the Mathematical Sciences. (2001). *The mathematical education of teachers: Issues in mathematics education* (Vol. 11). Providence, RI: American Mathematical Society.
- Crowley, M. L. (1987). The van Hiele model of the development of geometric thought. In M. M. Lindquist (Ed.), *Learning and*

- teaching geometry, K-12 (pp. 1-16). Reston, VA: National Council of Teachers of Mathematics.
- de Berg, K. C., & Greive, C. (1999). Understanding the siphon: An example of the development of pedagogical content knowledge using textbooks and the writings of early scientists. *Australian Science Teachers' Journal*, 45(4), 19-26.
- Delaney, S. (2008). *Adapting and using U.S. measures to study Irish teachers' mathematical knowledge for teaching*. Unpublished doctoral dissertation, University of Michigan, Ann Arbor.
- Grossman, P. L. (1990). *The making of a teacher: Teacher knowledge and teacher education*. New York: Teachers College Press.
- Grossman, P. L., Wilson, S. M., & Shulman, L. S. (1989). Teachers of substance: Subject matter knowledge for teaching. In M. Reynolds (Ed.), *The knowledge base for beginning teachers* (pp. 23-36). New York: Pergamon.
- Hapgood, S., Palincsar, A. S., Kucan, L., Gelpi-Lomangino, A., & Khasnabis, D. (2005, April). *Investigating a new measure of teachers' pedagogical content knowledge for teaching informational text comprehension*. Paper presented at the Annual Meeting of the American Educational Research Association, Montreal, Quebec, Canada.
- Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's Mathematics Professional Development Institutes. *Journal of Research in Mathematics Education*, 35, 330-351.
- Hill, H. C., Ball, D. L., & Schilling, S. G. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal*, 105(1), 11-30.
- Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Education Research Journal*, 42(2), 371-406.
- Hoyles, C., Noss, R., & Pozzi, S. (2001). Proportional reasoning in nursing practice. *Journal for Research in Mathematics Education*, 32, 4-27.
- Kent, P., Noss, R., Guile, D., Hoyles, C., & Bakker, A. (2007). Characterizing the use of mathematical knowledge in boundary crossing situations at work. *Mind, Culture, and Activity*, 14(1-2), 64-82.
- Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.
- Lowery, N. V. (2002). Construction of teacher knowledge in context: Preparing elementary teachers to teach mathematics and science. *School Science and Mathematics*, 102(2), 68-83.
- Magnusson, S., Krajcik, J., & Borko, H. (1999). Nature, sources and development of pedagogical content knowledge for science teaching. In J. Gess-Newsome & N. G. Lederman (Eds.), *Examining pedagogical content knowledge: The construct and its implications for science education* (pp. 95-132). Dordrecht, The Netherlands: Kluwer Academic.
- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.
- Marks, R. (1990). *Pedagogical content knowledge in elementary mathematics*. Unpublished doctoral dissertation, Stanford University, Palo Alto, CA.
- Moats, L. C. (1999). *Teaching reading is rocket science: What expert teachers of reading should know and be able to do*. Washington, DC: American Federation of Teachers.
- Millroy, W. L. (1992). An ethnographic study of the mathematical ideas of a group of carpenters. *Journal for Research in Mathematics Education* (Monograph Series No. 5).
- National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Washington, DC: U.S. Department of Education.
- Niess, M. L. (2005). Preparing teachers to teach science and mathematics with technology: Developing a technology pedagogical content knowledge. *Teaching and Teacher Education*, 21, 509-523.
- Noss, R., & Hoyles, C. (1996). The visibility of meanings: Designing for understanding the mathematics of banking. *International Journal of Computers for Mathematical Learning*, 1(1), 3-31.
- Phelps, G. (2008). *Just knowing how to read isn't enough! What teachers need to know about the content of reading*. Manuscript submitted for publication.
- Phelps, G., & Schilling, S. G. (2004). Developing measures of content knowledge for teaching reading. *Elementary School Journal*, 105(1), 31-48.
- Philipp, R. A., Cabral, C., & Schappelle, B. P. (2005). *IMAP CD-ROM: Integrating mathematics and pedagogy to illustrate children's reasoning* [Computer software]. Upper Saddle River, NJ: Pearson.
- Rovegno, I., Chen, W., & Todorovich, J. (2003). Accomplished teachers' pedagogical content knowledge of teaching dribbling to third grade children. *Journal of Teaching in Physical Education*, 22, 426-449.
- Schilling, S. G. (in press). The role of psychometric modeling in test validation for the MKT measures: An application of multidimensional item response theory. *Measurement: Interdisciplinary Research and Perspectives*.
- Schwab, J. J. (1978). Education and the structure of the disciplines. In I. Westbury & N. Wilkof (Eds.), *Science, curriculum, and liberal education* (pp. 167-183). Chicago: University of Chicago. (Original work published 1961)
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57, 1-22.
- Smith, E. L., & Anderson, C. W. (1984). Plants as producers: A case study of elementary science teaching. *Journal of Research in Science Teaching*, 21(7), 685-698.
- Stylianides, A. J., & Ball, D. L. (2004, April). *Studying the mathematical knowledge needed for teaching: The case of teachers' knowledge of reasoning and proof*. Paper prepared for the 2004 Annual Meeting of the American Educational Research Association, San Diego, CA.
- Thames, M. H. (2008). *A study of practice-based approaches for determining the mathematics that (K-8) teachers need to know*. Unpublished manuscript.
- Wilson, S., Shulman, L., & Richert, A. (1987). "150 different ways of knowing": Representations of knowledge in teaching. In J. Calderhead (Ed.), *Exploring teachers' thinking* (pp. 104-123). Eastbourne, UK: Cassell.
- Wilson, S. M. (1988). *Understanding historical understanding: Subject matter knowledge and the teaching of American history*. Unpublished doctoral dissertation, Stanford University, Palo Alto, CA.
- Wilson, S. M., & Wineburg, S. S. (1988). Peering at history through different lenses: The role of disciplinary perspectives in teaching history. *Teachers College Record*, 89(4), 525-539.
- Wineburg, S. (1990). *Historical problem-solving: A study of the cognitive processes used in the evaluation of documentary evidence*. Unpublished doctoral dissertation, Stanford University, Palo Alto, CA.

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